

# Transcritical bifurcation without parameters in memristive circuits\*

RICARDO RIAZA

Depto. de Matemática Aplicada a las TIC - ETSI Telecomunicación  
Universidad Politécnica de Madrid, Spain  
*ricardo.riaza@upm.es*

## Abstract

The transcritical bifurcation without parameters (TBWP) describes a stability change along a line of equilibria, resulting from the loss of normal hyperbolicity at a given point of such a line. Memristive circuits systematically yield manifolds of non-isolated equilibria, and in this paper we address a systematic characterization of the TBWP in circuits with a single memristor. To achieve this we develop two mathematical results of independent interest; the first one is an extension of the TBWP theorem to explicit ordinary differential equations (ODEs) in arbitrary dimension; the second result drives the characterization of this phenomenon to semiexplicit differential-algebraic equations (DAEs), which provide the appropriate framework for the analysis of circuit dynamics. In the circuit context the analysis is performed in graph-theoretic terms: in this setting, our first working scenario is restricted to passive problems (exception made of the bifurcating memristor), and in a second step some results are presented for the analysis of non-passive cases. The latter context is illustrated by means of a memristive neural network model.

**Keywords:** Manifold of equilibria, normal hyperbolicity, transcritical bifurcation without parameters, differential-algebraic equation, nonlinear circuit, memristor.

**AMS Subject Classification:** 34A09, 34C45, 34D35, 37G10, 94C05, 94C15.

## 1 Introduction

Memristors and other related electronic devices [10, 12, 41] are known to exhibit systematically manifolds of non-isolated equilibrium points. This is a consequence of the form of the voltage-current characteristic of memristive devices. As shown in [37], in the absence of certain configurations equilibrium manifolds of strictly locally passive memristive circuits are normally hyperbolic, that is, all remaining eigenvalues of the linearized vector field (except for those whose eigenvectors span the tangent space to the equilibrium manifold) are away from the imaginary axis. From a qualitative point of view it is

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therefore natural to examine what happens in the memristive circuit dynamics when the aforementioned passivity assumption does not hold.

This problem must be framed in the theory of *bifurcation without parameters* originally introduced in the seminal papers [14, 15, 16]; cf. also the recent book [25]. When normal hyperbolicity fails, a change in the local qualitative properties typically occurs along the equilibrium manifold, hence the “bifurcation without parameters” term. In this context, the most basic phenomenon is the transcritical bifurcation without parameters (TBWP throughout the whole manuscript), which describes the transition of one eigenvalue through the origin under certain local conditions on the vector field. Our purpose in this paper is to present a systematic circuit-theoretic characterization of this bifurcation for memristive circuits. Due to the systematic presence of non-isolated equilibria in memristive circuits, this is the most elementary phenomenon responsible for a stability loss in nonlinear circuits with memristors, and its analysis seems therefore to be very relevant for the development of the qualitative theory of memristive circuits.

This is actually a three-fold goal. First, the characterization of the TBWP in [14] (and also in [25]) is only addressed for two-dimensional dynamics. However, most nonlinear circuits involve a large number of dynamic variables and a two-dimensional model reduction is rarely feasible. For their results to apply to higher dimensional problems, the authors assume in [14, 25] that a prior reduction to a center two-dimensional manifold has been performed, but no explicit conditions paving the way for an appropriate reduction are given in arbitrary (finite) dimension. As a somewhat natural (yet not trivial) extension of their characterization we will present a TBWP theorem for explicit ODEs in  $\mathbb{R}^n$ , addressing the geometrical conditions which allow for a center manifold reduction in which the two-dimensional conditions of [14] do hold. This first goal is addressed in Section 2 (cf. Theorem 2).

Many nonlinear circuits do not admit a dynamical description in terms of an explicit ODE. This is clearly the case in large scale integration circuits, for which such a state space description in terms of an explicit ODE model is hardly automatable. For this reason, semistate models based on differential-algebraic equations (DAEs) are preferred instead [13, 18, 24, 34, 44]. Analytical results involving dynamical systems intended to apply to nonlinear circuit models should therefore be addressed to DAEs. Many qualitative investigations about nonlinear circuits require a prior reduction to an ODE model, involving unnecessarily restrictive hypotheses which, in addition, make the analysis more difficult (see e.g. [11] as a sample). In the DAE framework the approach is different: instead of trying to drive a model to the ODE context in order to apply a given known result, it is more convenient to extend such result to the DAE setting, allowing for a direct application to whatever semistate model. In this direction, our second goal is to drive the TBWP theorem to the semiexplicit DAE setting, a task which is accomplished in Section 3 and, specifically, in Theorem 3.

As indicated above, our third goal is to obtain a characterization of the TBWP for memristive circuits in circuit-theoretic terms. This means that the characterization should be stated in terms of the underlying circuit digraph and the electrical features of the de-

vices. This so-called *structural* approach has its roots in the state-space formulation problem (whose origins can be traced back to [5, 8]) and, more recently, has been successfully applied to the DAE index characterization of several nonlinear circuit models [13, 18, 20, 34, 42, 44, 45]. This approach makes it possible to directly transfer different analytical and qualitative results to circuit simulation programs. Allowed by the TBWP theorem for DAEs obtained in Section 3, such a characterization is detailed in Section 4 for circuits displaying a line of equilibria (that is, including exactly one memristor), under the assumption that the failing of a passivity assumption on this memristor is the one responsible for the loss of normal hyperbolicity: cf. Theorem 4. The analysis in this section extends some preliminary results presented in [17].

Section 5 discusses this phenomenon relaxing the passivity assumption on the remaining circuit devices, and includes an example coming from the theory of memristive neural networks. Finally, Section 6 compiles some concluding remarks.

## 2 The TBWP theorem for explicit ODEs

### 2.1 Two-dimensional dynamics

We begin by recalling the characterization of the TBWP in two-dimensional problems presented by Fiedler, Liebscher and Alexander in [14].

**Theorem 1** (Fiedler, Liebscher & Alexander, 2000). *Consider the system*

$$x' = \xi_1(x, y) \tag{1a}$$

$$y' = \xi_2(x, y), \tag{1b}$$

with  $\xi \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ , and assume that

1.  $\xi(x, 0) = 0$ ;
2.  $\frac{\partial \xi_2}{\partial y}(0, 0) = 0$ ;
3.  $\frac{\partial \xi_1}{\partial y}(0, 0) \neq 0$ ;
4.  $\frac{\partial^2 \xi_2}{\partial x \partial y}(0, 0) \neq 0$ .

Then (1) is locally orbitally  $C^1$ -equivalent to the normal form

$$x' = y \tag{2a}$$

$$y' = xy \tag{2b}$$

around the origin.

Needless to say, it is enough to assume that condition 1 holds for  $x$  sufficiently close to 0. Note that  $y = 0$  is a line of equilibria for both (1) and (2), and that  $\lambda = 0$  is an eigenvalue for the linearization of both systems at any  $(x, 0)$ . This zero eigenvalue becomes a double (index-two) one at  $(0, 0)$ , in a way such that the second system eigenvalue changes sign along the line of equilibria; specifically, this second eigenvalue is positive (resp. negative) if  $x > 0$  (resp.  $x < 0$ ) in the normal form (2). This means that the line of equilibria is normally hyperbolic for  $x \neq 0$ , and a stability change along the line of equilibria occurs as a result of the loss of normal hyperbolicity at the origin. This is the transcritical bifurcation without parameters.

## 2.2 The TBWP for explicit ODEs in $\mathbb{R}^n$

Theorem 1 can be extended to explicit ODEs in arbitrary (finite) dimension as follows. Mind a notational abuse used in different situations throughout the paper, namely writing both  $f(x)$  and  $f(x_1, \dots, x_n)$ , the latter standing for the (more cumbersome)  $f((x_1, \dots, x_n))$ . Obviously, the result below can be also stated for an open set  $\Omega \subseteq \mathbb{R}^n$  with  $0 \in \Omega$  or for a germ of a map at the origin.

**Theorem 2** (TBWP in  $\mathbb{R}^n$ ). *Assume that  $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$  verifies the following.*

1.  $f(x_1, 0, \dots, 0) = 0$ ;
2.  $f'(0)$  has a double index-two zero eigenvalue, and  $\operatorname{Re} \lambda \neq 0$  for the remaining ones;
3.  $f''(0)pq \notin \operatorname{im} f'(0)$ , if  $p \in \ker f'(0) - \{0\}$  and  $q \in \ker (f'(0))^2 - \ker f'(0)$ .

*Then there exists a local, two-dimensional,  $C^2$  center manifold, where the reduced dynamics admits a description in local coordinates of the form  $u' = \xi(u)$ , with  $\xi$  verifying the conditions of Theorem 1 in  $u^* = (0, 0)$ .*

**Proof.** The proof relies on the fact that the linear transformation driving the linear part to Jordan form leaves the line of equilibria invariant; note also that condition 3 captures the geometric (transversality) hypothesis which extends to higher-dimensional contexts the second-order condition in Theorem 1. For better clarity we proceed in numbered steps.

1. Let  $P$  be a matrix that drives  $f'(0)$  to Jordan form:

$$\tilde{J} = P^{-1}f'(0)P = \begin{pmatrix} J_0 & 0 \\ 0 & J_h \end{pmatrix}, \quad J_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J_h \text{ hyperbolic.}$$

2. Under the change of coordinates  $x = Py$ , the system  $x' = f(x)$  is transformed into  $y' = \tilde{f}(y) = P^{-1}f(Py)$ , which reads as

$$\begin{aligned} u' &= \tilde{f}_1(u, v) = J_0 u + \eta(u, v) \\ v' &= \tilde{f}_2(u, v) = J_h v + \gamma(u, v) \end{aligned}$$

with  $y = (u, v)$ ,  $u \in \mathbb{R}^2$ ,  $v \in \mathbb{R}^{n-2}$  and  $\eta'(0, 0) = 0$ ,  $\gamma'(0, 0) = 0$ .

3. Now, from the fact that  $\tilde{p} \in \ker \tilde{f}'(0) \Leftrightarrow p = P\tilde{p} \in \ker f'(0)$  it follows that  $P$  leaves the space  $\ker f'(0) = \text{span}\{e_1\}$  (with  $e_1 = (1, 0, \dots, 0)$ ) invariant, and therefore the equilibrium line of  $f$  is  $(u_1, 0, \dots, 0)$ .

4. Using, analogously, the properties  $\tilde{q} \in \ker (\tilde{f}'(0))^2 \Leftrightarrow q = P\tilde{q} \in \ker (f'(0))^2$  and  $\tilde{w} \in \text{im } \tilde{f}'(0) \Leftrightarrow w = P\tilde{w} \in \text{im } f'(0)$ , the condition

$$f''(0)pq \notin \text{im } f'(0) \text{ with } p \in \ker f'(0) - \{0\} \text{ and } q \in \ker (f'(0))^2 - \ker f'(0)$$

yields

$$\tilde{f}''(0)\tilde{p}\tilde{q} \notin \text{im } \tilde{f}'(0), \text{ with } \tilde{p} \in \ker \tilde{f}'(0) - \{0\} \text{ and } \tilde{q} \in \ker (\tilde{f}'(0))^2 - \ker \tilde{f}'(0)$$

and, in turn, this leads to

$$\frac{\partial^2 \eta_2}{\partial u_1 \partial u_2}(0, 0) \neq 0. \quad (3)$$

5. The system  $y' = \tilde{f}(y)$  admits a local center manifold of the form  $v = \zeta(u)$ , with  $\zeta(0) = 0$ ,  $\zeta'(0) = 0$  (see e.g. [9, 32, 48]). The dynamics on this manifold reads as

$$u' = \xi(u) = J_0 u + \eta(u, \zeta(u)).$$

6. Locally, the curve of equilibria must belong to the center manifold [9, 48], and this yields condition 1 of Theorem 1, that is,  $\xi(u_1, 0) = 0$ . Additionally, the form of  $J_0$  renders conditions 2-3 trivial, and (3) yields condition 4, that is,

$$\frac{\partial^2 \xi_2}{\partial u_1 \partial u_2}(0, 0) \neq 0,$$

since  $\eta'(0, 0) = 0$ ,  $\zeta'(0, 0) = 0$ . This completes the proof. □

**Remark.** Geometrically, condition 3 expresses the transversality at  $x^*$  of the center manifold and the so-called singular manifold

$$\{x \in \mathbb{R}^n / \det f'(x) = 0\}, \quad (4)$$

as a consequence of the following well-known property from matrix analysis (we omit the proof; find details e.g. in [36]).

**Lemma 1.** *If  $H \in C^1(\mathbb{R}^m, \mathbb{R}^{n \times n})$ ,  $\text{rk } H(x^*) = n - 1$  and  $p \in \ker H(x^*) - \{0\}$ , then*

$$(H'(x^*)q)p \notin \text{im } H(x^*) \Leftrightarrow (\det H)'(x^*)q \neq 0.$$

Lemma 1 implies in particular that, if  $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$  and  $\text{rk } f'(x^*) = n - 1$ , then

$$f''(x^*)pq \notin \text{im } f'(x^*) \Leftrightarrow (\det f')'(x^*)q \neq 0,$$

a condition which, in the setting of Theorem 2, expresses the transversal intersection of the direction spanned by the generalized eigenvector  $q$  (hence of the center manifold itself) and the singular manifold (4), as indicated in the Remark above. For later use, we also note that for a single-parameter valued matrix map  $H \in C^1(\mathbb{R}, \mathbb{R}^{n \times n})$  with  $H'(\lambda^*) = n - 1$  we have

$$H'(\lambda^*)p \notin \text{im } H(\lambda^*) \Leftrightarrow (\det H)'(\lambda^*) \neq 0.$$

Lemma 1 also shows that condition 3 does not depend on the choice of  $q$ , since  $\ker f'(0)$  is tangent to the singular manifold, so that  $f''(0)pp \in \text{im } f'(0)$  and then, for  $\hat{q} = \alpha q + \beta p$  with  $\alpha \neq 0$ , we have

$$f''(0)p\hat{q} = (\alpha f''(0)pq + \beta f''(0)pp) \notin \text{im } f'(0) \Leftrightarrow f''(0)pq \notin \text{im } f'(0).$$

Note also that the form of  $\ker f'(0) = \text{span}\{e_1\}$  makes it possible to simplify the statement of condition 3 to

$$f_{x_1x}(0)q \notin \text{im } f'(0),$$

where  $f_{x_1x}$  denotes the matrix of partial derivatives  $\left( \frac{\partial^2 f_i}{\partial x_1 \partial x_j} \right)$ .

Finally, from the Šořitaišvili-Palmer Theorem [31], it follows that the normal form for the TBWP in  $\mathbb{R}^n$  is

$$\begin{aligned} x' &= y \\ y' &= xy \\ v' &= J_h v. \end{aligned}$$

Certainly, the form of the latter equation may be further simplified to that of a standard node or saddle point, depending on the inertia of  $J_h$ .

### 3 TBWP in semiexplicit DAEs

Along the route indicated in Section 1, we extend below the TBWP to the setting of semiexplicit DAEs.

**Theorem 3** (TBWP in semiexplicit index-one DAEs). *Let  $h \in C^2(\mathbb{R}^{r+p}, \mathbb{R}^r)$ ,  $g \in C^2(\mathbb{R}^{r+p}, \mathbb{R}^p)$ , and consider the system*

$$y' = h(y, z) \tag{5a}$$

$$0 = g(y, z). \tag{5b}$$

Write  $E = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ ,  $F = (h, g)$ . Assume that  $g_z(0, 0)$  is non-singular and that

1.  $h(y_1, 0, 0) = 0, g(y_1, 0, 0) = 0$ ;
2. the matrix pencil  $\lambda E - F'(0, 0)$  has a double index-two zero eigenvalue, and  $\operatorname{Re} \lambda \neq 0$  for the remaining eigenvalues;
3.  $F''(0, 0)\overline{p}\overline{q} \notin \operatorname{im} F'(0, 0)$ , where

$$\overline{p} \in \ker F'(0, 0) - \{0\}, \quad \overline{q} \in \ker (F'(0, 0))^2 - \ker F'(0, 0). \quad (6)$$

Then, there exists an invariant, two-dimensional,  $C^2$  submanifold of  $g(y, z) = 0$  where the dynamics admits a local description of the form  $u' = \xi(u)$  with  $\xi$  satisfying the conditions of Theorem 1 at the origin.

Before proceeding with the proof we present some auxiliary results.

**Lemma 2** (Schur). *Let  $D$  be a non-singular matrix and*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (M/D) = A - BD^{-1}C, \quad (7)$$

with  $A$  (hence  $M$ ) square. Then  $\det M = \det(M/D) \det D$  and  $\operatorname{cork} M = \operatorname{cork}(M/D)$ .

We will make use of this Lemma at several points in our analysis, mostly with

$$M = F'(0, 0) = \begin{pmatrix} h_y(0, 0) & h_z(0, 0) \\ g_y(0, 0) & g_z(0, 0) \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \quad (8)$$

The proof of Theorem 3 will be based on checking conditions 1-3 of Theorem 2 for the reduced dynamics of the DAE (5) on the solution manifold (5b). Conditions 1 and 2 will be derived in a more or less straightforward manner; condition 3 is not trivial, though. Remember that the goal is to state the conditions in terms of the original problem setting, that is, in terms of  $h$  and  $g$  (that is, of  $F$ ), as it is done in our statement of condition 3. But note that it is the matrix pencil  $\lambda E - F'(0, 0)$  (and not the matrix  $F'(0, 0)$ ) the one that is assumed to have a double, index-two zero eigenvalue; this means that it is not obvious that there should exist an  $\overline{q}$  satisfying the requirement depicted in (6) in light of the previous hypotheses. As a cautionary example consider, for instance, the Schur reduction of

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad (M/D) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and note that  $\lambda = 0$  is a double eigenvalue for  $(M/D)$  but a simple one for  $M$ ; no generalized eigenvector exists in this case for  $M$ . This cannot occur in the setting of Theorem 3 (that is, there will indeed exist a  $\overline{q}$  satisfying the condition in (6)) because of item (c) of Lemma 3 below.

**Lemma 3.** *Given  $M$  and  $(M/D)$  in (7), consider the operators  $L$  and  $T : \mathbb{R}^r \rightarrow \mathbb{R}^{r+p}$  defined by*

$$Lu = \begin{pmatrix} u \\ -D^{-1}Cu \end{pmatrix}, \quad Tu = \begin{pmatrix} u \\ 0 \end{pmatrix}.$$

*Then*

- (a)  $\bar{p} \in \ker M \Leftrightarrow \bar{p} = Lp$ , with  $p \in \ker (M/D)$ ;
- (b)  $w \in \operatorname{im} (M/D) \Leftrightarrow Tw \in \operatorname{im} M$ . Actually,  $Tw = M\bar{u} \Leftrightarrow \bar{u} = Lu$ , with  $w = (M/D)u$ ;
- (c) if  $\ker M \subseteq \operatorname{im} T = \mathbb{R}^r \times \{0\}$ , then  $\bar{q} \in \ker M^2 \Leftrightarrow \bar{q} = Lq$ , with  $q \in \ker (M/D)^2$ .

**Proof.** Items (a) and (b) are immediate in light of the definition of  $(M/D)$  in (7). Regarding (c), let  $\bar{q} \in \ker M^2$  and denote  $\bar{w} = M\bar{q} \in \ker M \cap \operatorname{im} M$ . Then:

- (i) since  $\bar{w} \in \ker M$ , then  $\bar{w} = Lw$  with  $w \in \ker (M/D)$  because of (a);
- (ii) owing to the hypothesis  $\ker M \subseteq \operatorname{im} T$ , necessarily  $\bar{w} = Tu$  for a certain  $u$ ; additionally, because of the form of  $L$  and  $T$  it follows that  $u = w$  and then  $\bar{w} = Lw = Tw$ ;
- (iii) the condition  $\bar{w} = Tw = M\bar{q}$  implies, because of (b), that  $\bar{q} = Lq$ , with  $w = (M/D)q$ ;
- (iv) finally, since  $w \in \ker (M/D)$  (cf. (i)), it follows that  $q \in \ker (M/D)^2$ .

The converse result in (c) is entirely analogous and the proof is left to the reader. □

With  $M$  as defined in (8), then the linear operator  $L$  is the differential at the origin of the parameterization  $y \rightarrow (y, \psi(y))$  of the manifold  $\mathcal{M}$  in (5b), with  $\psi$  given by the implicit function theorem, and therefore  $L$  will define an isomorphism  $\mathbb{R}^r \rightarrow \ker (C \ D) = T_{(0,0)}\mathcal{M}$ . Item (a) in Lemma 3 expresses that  $L$  also induces an isomorphism between the spaces  $\ker (M/D) \rightarrow \ker M \subseteq T_{(0,0)}\mathcal{M}$ . Moreover, in the scenario assumed in (c), one can check that  $\ker M^2 \subseteq T_{(0,0)}\mathcal{M}$  and that  $L$  also induces an isomorphism  $\ker (M/D)^2 \rightarrow \ker M^2$ .

For later use, we also note that a coordinate description  $\alpha$  of a map  $\beta$  defined on  $\mathcal{M}$ , that is, a relation of the form  $\alpha(y) = \beta(y, \psi(y))$ , implies  $\alpha'(0) = \beta'(0, \psi(0))L$ . We will make use of this remark in the final step of the proof of Theorem 3.

### Proof of Theorem 3.

1. The hypothesis that  $g_z(0,0)$  is non-singular implies, by the implicit function theorem, that  $\mathcal{M} \equiv g = 0$  is locally a manifold that can be described by  $z = \psi(y)$ . The goal is then to apply Theorem 2 to the reduced system

$$y' = f(y) = h(y, \psi(y)). \tag{9}$$

Specifically, we need to check that the requirements imposed on  $F = (h, g)$  yield the conditions 1-3 in Theorem 2.

2. The first condition holds trivially, since

$$f(y_1, 0) = h(y_1, 0, \psi(y_1, 0)) = h(y_1, 0, 0) = 0,$$



where the second identity is due to the fact that  $\psi(y_1, 0) = 0$  because  $g(y_1, 0, 0) = 0$ .

3. The second condition also follows easily from the implicit function theorem, since the linearization of the reduced system (9) at the origin is

$$f'(0) = h_y(0, 0) + h_z(0, 0)\psi'(0) = h_y(0, 0) - h_z(0, 0)(g_z(0, 0))^{-1}g_y(0, 0),$$

and the spectrum of  $f'(0) = A - BD^{-1}C$  equals that of the matrix pencil  $\lambda E - M$ , as an immediate consequence of Schur's lemma:

$$\begin{aligned} \det(\lambda E - M) &= \det \left( \lambda \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) \\ &= \det \begin{pmatrix} \lambda I_r - A & -B \\ -C & -D \end{pmatrix} = \det(\lambda I_r - (A - BD^{-1}C)) \det(-D). \end{aligned}$$

4. The only non-immediate step consists in checking that

$$F''(0, 0)\overline{p}\overline{q} \notin \text{im } F'(0, 0),$$

where  $\overline{p} \in \ker F'(0, 0) - \{0\}$ ,  $\overline{q} \in \ker (F'(0, 0))^2 - \ker F'(0, 0)$ , implies

$$f''(0)pq \notin \text{im } f'(0),$$

with  $p \in \ker f'(0) - \{0\}$  and  $q \in \ker (f'(0))^2 - \ker f'(0)$ . Note that item (c) of Lemma 3 applies because  $\ker F'(0, 0) = \text{span}\{e_1\}$ , and then  $\overline{q}$  yields a generalized eigenvector  $q$  of  $f'(0)$ , with  $\overline{q} = Lq$ .

It then suffices to use the characterization

$$F''(0, 0)\overline{p}\overline{q} \notin \text{im } F'(0, 0) \Leftrightarrow (\det F')'(0, 0)\overline{q} \neq 0,$$

following from Lemma 1 because, differentiating  $\det F' = \det g_z \det(F'/g_z)$  and using the fact that  $\det(F'/g_z)(0, 0) = \det(f')(0) = 0$  (because  $\lambda = 0$  is an eigenvalue), we have

$$(\det F')'(0, 0) = \det g_z(0, 0)(\det(F'/g_z))'(0, 0);$$

additionally, since  $\det g_z(0, 0) \neq 0$ , it follows that

$$(\det F')'(0, 0)\overline{q} \neq 0 \Leftrightarrow (\det(F'/g_z))'(0, 0)Lq \neq 0 \Leftrightarrow (\det(f'))'(0)q \neq 0,$$

because  $\det f'(y) = \det(F'/g_z)(y, \psi(y)) \Rightarrow (\det(f'))'(0) = (\det(F'/g_z))'(0, 0)L$  as indicated right before the proof of Theorem 3. Finally,

$$(\det(f'))'(0)q \neq 0 \Leftrightarrow f''(0)pq \notin \text{im } f'(0)$$

and the proof is complete. □

**Remark.** As in the explicit ODE case, condition 3 can be recast as

$$F_{x_1x}(0, 0)\overline{q} \notin \text{im } F'(0, 0),$$

where  $x = (y, z)$  and  $F_{x_1x}$  stands for the matrix of partial derivatives  $\left( \frac{\partial^2 F_i}{\partial x_1 \partial x_j} \right)$ .

## 4 TBWP in memristive circuits

The memristor (an abbreviation for *memory-resistor*) is a new electronic device governed by a nonlinear,  $C^1$  flux-charge relation of the form  $\varphi = \phi(q)$ . The existence of such a device was predicted for symmetry reasons by Chua in 1971 [10], but it was not until 2008 that it began to attract considerable attention. The reason for this was the report in [41] of the design of a nanometric memristor by the HP company. The key aspect of this device is that, by differentiation of the aforementioned constitutive relation, one gets the voltage-current relation

$$v = M(q)i, \quad (10)$$

with  $M(q) = \phi'(q)$ . For later use, we will assume that  $\phi$  is a  $C^2$  map. Note that in (10) the “resistance” (or, better, *memristance*)  $M$  depends on  $q(t) = \int_{-\infty}^t i(\tau)d\tau$ , so that the device somehow keeps track of its own history (hence the memory-resistor name). A great amount of research has been directed to this and other related devices since 2008; cf. [1, 12, 19, 22, 23, 26, 28, 29, 33, 35, 37, 43] as a sample of literature.

It is easy to check that memristors systematically yield manifolds of non-isolated equilibrium points; find details below. In order to focus on problems with *lines* of equilibria, we will restrict our attention to circuits including a single memristor, besides capacitors, inductors, resistors, and (independent) voltage and current sources. Capacitors, inductors and resistors may be nonlinear, and they will respectively be assumed to be defined by a voltage-dependent capacitance matrix  $C(v_c)$ , a current-dependent inductance matrix  $L(i_l)$ , and a current-controlled  $C^1$  description  $v_r = \gamma(i_r)$  in the case of resistors; for later use we denote the resistance matrix  $\gamma'(i_r)$  as  $R(i_r)$ . All three matrices need not be diagonal, allowing for the presence of coupling effects in the corresponding sets of devices. Capacitors, inductors and resistors are said to be strictly locally passive at a given operating point if the corresponding characteristic matrix, that is,  $C(v_c)$ ,  $L(i_l)$  or  $R(i_r)$ , is positive definite (a square matrix  $P$  is positive definite if it verifies  $v^\top P v > 0$  for non-vanishing real vectors  $v$ ; note that we do not require these matrices to be symmetric). We assume the circuit to be autonomous, that is, sources are DC ones or, in mathematical terms, they take constant values (grouped together in vectors  $V$  and  $I$ ).

In this context, the circuit equations can be modeled by the differential-algebraic system (see e.g. [34])

$$q'_m = i_m \quad (11a)$$

$$C(v_c)v'_c = i_c \quad (11b)$$

$$L(i_l)i'_l = v_l \quad (11c)$$

$$0 = B_m M(q_m) i_m + B_c v_c + B_l v_l + B_r \gamma(i_r) + B_u V + B_j v_j \quad (11d)$$

$$0 = Q_m i_m + Q_c i_c + Q_l i_l + Q_r i_r + Q_u i_u + Q_j I, \quad (11e)$$

where the subscripts  $m, c, l, r, u, j$  are used for memristors, capacitors, inductors, resistors, voltage sources and current sources, respectively. It is worth emphasizing that (11d) and (11e) express Kirchhoff voltage and current laws in terms of the so-called loop and cutset matrices  $B$  and  $Q$  (cf. the Appendix and [6, 34, 44]); we split these matrices as

$B = (B_m \ B_c \ B_l \ B_r \ B_u \ B_j)$  and  $Q = (Q_m \ Q_c \ Q_l \ Q_r \ Q_u \ Q_j)$ , where  $B_m$  (resp.  $B_c$ ,  $B_l$ ,  $B_r$ ,  $B_u$ ,  $B_j$ ) corresponds to the columns of  $B$  accommodating memristors (resp. capacitors, inductors, resistors, voltage sources, current sources), and the same notational convention applies to the cutset matrix.

By denoting  $y = (q_m, v_c, i_l)$ ,  $z = (i_m, i_c, v_l, i_r, v_j, i_u)$ , the DAE (11) takes the form

$$\begin{aligned} E(y)y' &= h(y, z) \\ 0 &= g(y, z) \end{aligned}$$

and equilibria are defined by the pair of conditions  $h(y, z) = 0$ ,  $g(y, z) = 0$ , that is,

$$i_m = i_c = v_l = 0 \quad (12a)$$

$$B_c v_c + B_r \gamma(i_r) + B_u V + B_j v_j = 0 \quad (12b)$$

$$Q_l i_l + Q_r i_r + Q_u i_u + Q_j I = 0. \quad (12c)$$

Note that the variable  $q_m$  is not at all involved in (12). This means that, necessarily, no equilibrium point may be isolated, since the variable  $q_m$  unfolds any given equilibrium point to a line (or even a higher dimensional set) of equilibria. In our working setting equilibria will actually define a line, as a consequence of the condition  $\text{cork } F' = 1$  shown within the proof of Theorem 4.

The previous remarks drive the stability analysis of equilibria in memristive circuits to the mathematical context considered in [4, 14, 15, 16, 25]. In this setting, the existence of an  $m$ -dimensional manifold of equilibria implies that at least  $m$  eigenvalues of the linearization of the vector field at any of these equilibria are null. The manifold is then said to be *normally hyperbolic* (locally around such an equilibrium) if the remaining eigenvalues are not in the imaginary axis. The failing of the normal hyperbolicity requirement typically yields a bifurcation without parameters [14, 15, 16, 25], where the qualitative properties of the local phase portrait change.

In [37] one can find graph-theoretic conditions under which any manifold of equilibria of a strictly locally passive memristive circuit is guaranteed to be normally hyperbolic. It is therefore natural to address what happens when the passivity assumption does not hold; allowed by Theorem 3, Theorem 4 below answers this question, again in circuit-theoretic terms, for circuits with one memristor which becomes locally active at a given operating point, yielding a transcritical bifurcation without parameters. Note that in the statement of Theorem 4, a VMC-loop is a loop composed only of voltage sources, memristors and/or capacitors. ILC-cutsets, VML-loops, etc. are defined analogously.

**Theorem 4** (TBWP in memristive circuits). *Consider a nonlinear circuit with a single memristor, modeled by (11). Fix an equilibrium point  $(q_m^*, v_c^*, i_l^*, i_m^*, i_c^*, v_l^*, i_r^*, v_j^*, i_u^*)$  (with  $i_m^* = 0$ ,  $i_c^* = 0$ ,  $v_l^* = 0$ ), and assume that the following conditions hold.*

1. *The circuit displays neither VMC-loops nor ILC-cutsets.*
2. *There is a unique VML-loop, which includes the memristor and at least one inductor.*

3. The capacitance, inductance and resistance matrices  $C(v_c^*)$ ,  $L(i_l^*)$ ,  $R(i_r^*) = \gamma'(i_r^*)$  are positive definite, with  $C(v_c^*)$  and  $L(i_l^*)$  symmetric; additionally,  $M(q_m^*) = 0$  and  $M'(q_m^*) \neq 0$ .

Then the circuit undergoes a transcritical bifurcation without parameters at the aforementioned equilibrium point; moreover, near this bifurcating equilibrium, all eigenvalues of the linearization (but the null one) have negative real part in the region where  $M(q_m) > 0$ , whereas a single (real) eigenvalue becomes positive at points where  $M(q_m) < 0$ .

In the proof of Theorem 4 we will make use of some graph-theoretic results which are compiled in advance. Proofs of these auxiliary results can be found in [3, 6, 34, 37].

**Lemma 4.** Let  $B_i$  and  $Q_i$  denote, for  $i = 1, 2, 3$ , the submatrices of  $B$  and  $Q$  defined by three pairwise-disjoint branch sets  $K_1, K_2, K_3$  of a given directed graph. If  $P$  is a positive definite matrix, then

$$\ker \begin{pmatrix} B_1 & 0 & B_3 P \\ 0 & Q_2 & Q_3 \end{pmatrix} = \ker B_1 \times \ker Q_2 \times \{0\}.$$

The same terminological convention is used in Lemma 5 below. By a  $K_i$ -cutset (resp. loop) we mean a cutset (resp. loop) defined only by branches belonging to  $K_i$ .

**Lemma 5.** The identity  $\ker B_1 = \{0\}$  (resp.  $\ker Q_2 = \{0\}$ ) holds if and only if the digraph has no  $K_1$ -cutsets (resp.  $K_2$ -loops).

The proof of Theorem 4 below is based on Theorem 3, which for simplicity was stated under the assumption that the bifurcating equilibrium is located at the origin. Obviously, we can make use of this result at a generic equilibrium  $(y^*, z^*)$  (with  $y^* = (q_m^*, v_c^*, i_l^*)$  and  $z^* = (i_m^*, i_c^*, v_l^*, i_r^*, v_j^*, i_u^*)$ ) and we will do so without further explicit mention.

**Proof of Theorem 4.** Note first that the strict passivity assumption on  $C(v_c^*)$  and  $L(i_l^*)$  makes these matrices non-singular, and therefore the maps  $h$  and  $g$  from (5) have (at least locally) the form

$$\begin{aligned} h(y, z) &= \begin{pmatrix} i_m \\ (C(v_c))^{-1} i_c \\ (L(i_l))^{-1} v_l \end{pmatrix} \\ g(y, z) &= \begin{pmatrix} B_m M(q_m) i_m + B_c v_c + B_l v_l + B_r \gamma(i_r) + B_u V + B_j v_j \\ Q_m i_m + Q_c i_c + Q_l i_l + Q_r i_r + Q_u i_u + Q_j I \end{pmatrix}, \end{aligned}$$

where we are denoting  $y = (q_m, v_c, i_l)$ ,  $z = (i_m, i_c, v_l, i_r, v_j, i_u)$ .

1. The matrix of partial derivatives  $g_z(y^*, z^*)$  is (using  $i_m^* = 0$ ,  $M(q_m^*) = 0$ )

$$g_z(y^*, z^*) = \begin{pmatrix} 0 & 0 & B_l & B_r R(i_r^*) & B_j & 0 \\ Q_m & Q_c & 0 & Q_r & 0 & Q_u \end{pmatrix},$$

which is invertible in light of Lemmas 4 and 5, since  $R(i_r^*)$  is positive definite and there are neither IL-cutsets (which are a particular instance of an ILC-cutset) nor VMC-loops.

2. Denoting  $y = (q_m, \tilde{y})$ , the conditions  $h(q_m, \tilde{y}^*, z^*) = 0$ ,  $g(q_m, \tilde{y}^*, z^*) = 0$  (arising in condition 1 of Theorem 3) are trivially met (cf. (12)).

3. Condition 2 of Theorem 3 involves a matrix pencil spectrum which is given by the determinant of

$$\begin{pmatrix} \lambda & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda I_c & 0 & 0 & -(C(v_c^*))^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda I_l & 0 & 0 & -(L(i_l^*))^{-1} & 0 & 0 & 0 \\ 0 & -B_c & 0 & 0 & 0 & -B_l & -B_r R(i_r^*) & -B_j & 0 \\ 0 & 0 & -Q_l & -Q_m & -Q_c & 0 & -Q_r & 0 & -Q_u \end{pmatrix}, \quad (14)$$

and which can be written as  $\lambda d(\lambda)$  with  $d(\lambda) = \det K(\lambda)$ , provided that  $K(\lambda)$  is the bottom-right submatrix of (14) (that is, the one obtained after removing the top row and the left column).

The eigenvalue  $\lambda = 0$  being a double one amounts to  $d(0) = 0$  (ie. to  $K(0)$  being singular) with  $d'(0) \neq 0$ . Additionally, provided that  $\text{cork } K(0) = 1$  (a condition which will be proved to hold), then the condition  $d'(0) \neq 0$  is equivalent (cf. Lemma 1) to  $K'(0)p \notin \text{im } K(0)$  for  $p \in \ker K(0) - \{0\}$ . Finally, for the (double) zero eigenvalue to be index-two it will be enough to show that  $\text{cork } F'(y^*, z^*) = 1$ . We examine this set of conditions in items 4, 5 and 6 below.

4. The matrix  $K(0)$  reads as

$$\begin{pmatrix} 0 & 0 & 0 & -(C(v_c^*))^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(L(i_l^*))^{-1} & 0 & 0 & 0 \\ -B_c & 0 & 0 & 0 & -B_l & -B_r R(i_r^*) & -B_j & 0 \\ 0 & -Q_l & -Q_m & -Q_c & 0 & -Q_r & 0 & -Q_u \end{pmatrix}$$

and, following Schur's lemma (Lemma 2), this matrix is easily seen to have the same corank as

$$\begin{pmatrix} B_c & 0 & 0 & B_r R(i_r^*) & B_j & 0 \\ 0 & Q_l & Q_m & Q_r & 0 & Q_u \end{pmatrix}.$$

From Lemmas 4 and 5, the positive definiteness of  $R(i_r^*)$ , the absence of IC-cutsets and the existence of a unique VML-loop, it follows that  $\text{cork } K(0) = 1$ , with

$$\ker K(0) = \text{span}\{(0, p_l, p_m, 0, 0, 0, 0, p_u)\}$$

where  $(p_l, p_m, p_u) \in \ker (Q_l \ Q_m \ Q_u)$  (and note, for later use, that  $p_l \neq 0$ ).

5. The condition  $K'(0)p \notin \text{im } K(0)$  is

$$\begin{pmatrix} 0 \\ p_l \\ 0 \\ 0 \end{pmatrix} \notin \text{im} \begin{pmatrix} 0 & 0 & 0 & -(C(v_c^*))^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(L(i_l^*))^{-1} & 0 & 0 & 0 \\ -B_c & 0 & 0 & 0 & -B_l & -B_r R(i_r^*) & -B_j & 0 \\ 0 & -Q_l & -Q_m & -Q_c & 0 & -Q_r & 0 & Q_u \end{pmatrix}.$$

Assuming that this condition does not hold, we would have a solution for

$$B_c u_1 - B_l L(i_l^*) p_l + B_r R(i_r^*) u_6 + B_j u_7 = 0.$$

Together with the identity  $Q_l p_l + Q_m p_m + Q_u p_u = 0$  and the orthogonality of the so-called cut and cycle spaces  $\ker B$ ,  $\ker Q$  (cf. [6]), we would get  $p_l^\top L(i_l^*) p_l = 0$  and therefore  $p_l = 0$  (since  $L(i_l^*)$  is positive definite), against the fact that  $p_l \neq 0$ .

Hence,  $K'(0)p \notin \text{im } K(0)$  for  $p \in \ker K(0) - \{0\}$ , and as indicated above this implies that  $\lambda = 0$  is a double eigenvalue.

6. The matrix  $F'(y^*, z^*)$  is

$$F'(y^*, z^*) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (C(v_c^*))^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (L(i_l^*))^{-1} & 0 & 0 & 0 \\ 0 & B_c & 0 & 0 & 0 & B_l & B_r R(i_r^*) & B_j & 0 \\ 0 & 0 & Q_l & Q_m & Q_c & 0 & Q_r & 0 & Q_u \end{pmatrix}$$

and, via Schur's lemma,  $\text{cork } F'(y^*, z^*) = 1$  follows again from Lemmas 4 and 5, which imply that

$$\ker \begin{pmatrix} B_c & 0 & B_r R(i_r^*) & B_j & 0 \\ 0 & Q_l & Q_r & 0 & Q_u \end{pmatrix} = \{0\}.$$

As indicated above, this implies that the double zero eigenvalue is indeed index two.

7. The fact that all non-vanishing eigenvalues ( $\lambda \neq 0$ ) have non-zero real part follows from the eigenvalue-eigenvector equations of the pencil, which can be written as

$$\begin{aligned} B_c u_c + \lambda B_l L(i_l^*) w_l + B_r R(i_r^*) w_r + B_j u_j &= 0 \\ \lambda Q_c C(v_c^*) u_c + Q_l w_l + Q_r w_r + Q_m w_m + Q_u w_u &= 0 \end{aligned}$$

together with  $\lambda \sigma_m = w_m$ ,  $w_c = \lambda C(v_c^*) u_c$ ,  $u_l = \lambda L(i_l^*) w_l$ . By taking conjugate transposes and using the orthogonality of the cut and cycle spaces, we derive

$$(\text{Re } \lambda) (u_c^* C(v_c^*) u_c + w_l^* L(i_l^*) w_l) + w_r^* \frac{R(i_r^*) + (R(i_r^*))^\top}{2} w_r = 0. \quad (15)$$

Now, if  $\text{Re } \lambda = 0$ , the positive definiteness of  $R(i_r^*)$  implies  $w_r = 0$ , and then

$$B_c u_c + \lambda B_l L w_l + B_j u_j = 0 \quad (16a)$$

$$\lambda Q_c C u_c + Q_l w_l + Q_m w_m + Q_u w_u = 0. \quad (16b)$$

But in this setting the hypothesis that there are no ILC-cutsets implies, in light of (16a),  $u_c = 0$  (and then  $w_c = 0$ ),  $w_l = 0$  (so that  $u_l = 0$ ) and  $u_j = 0$ . Additionally, the absence of VM-loops and (16b) would then imply  $w_m = 0$  (and then  $\sigma_m = 0$ ) and  $w_u = 0$ ; this would yield a vanishing eigenvector, which is a contradiction in terms.

8. In order to check that condition 3 of Theorem 3 holds, we take  $\bar{q}$  from the requirement  $F'(y^*, z^*)\bar{q} \in \ker F'(y^*, z^*) - \{0\}$ , which gives  $\bar{q}$  the form

$$\bar{q} = (\bar{q}_1, 0, \bar{q}_3, \bar{q}_4, 0, 0, 0, 0, \bar{q}_9), \text{ with } (\bar{q}_3, \bar{q}_4, \bar{q}_9) \in \ker (Q_l \ Q_m \ Q_u) - \{0\}$$

and, in particular,  $\bar{q}_4 \neq 0$ . The condition  $F''(y^*, z^*)\bar{p}\bar{q} \notin \text{im } F'(y^*, z^*)$  then reads as

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ B_m M'(q^*)\bar{q}_4 \\ 0 \end{pmatrix} \notin \text{im} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (C(v_c^*))^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (L(i_l^*))^{-1} & 0 & 0 & 0 \\ 0 & B_c & 0 & 0 & 0 & B_l & B_r R(i_r^*) & B_j & 0 \\ 0 & 0 & Q_l & Q_m & Q_c & 0 & Q_r & 0 & Q_u \end{pmatrix}.$$

Again, assuming that this condition is not met, we would derive the existence of a non-trivial solution for

$$B_c u_2 - B_m M'(q^*)\bar{q}_4 + B_r R(i_r^*)u_7 + B_i u_8 = 0,$$

but the presence of a VML-loop with the memristor rules out, because of the colored branch theorem (according to which, in a three-color graph with just one blue branch, this branch cannot form simultaneously a loop exclusively with green branches and a cutset only with red branches; cf. [27, 46]), the existence of any CMRI-cutset including the memristor. This would yield  $M'(q^*)\bar{q}_4 = 0$ , against the fact that  $M'(q^*) \neq 0 \neq \bar{q}_4$ .

9. Finally, the proof that all eigenvalues (but the null one) have negative real part at points of the equilibrium half-line where  $M(q_m) > 0$  (and, certainly, on a neighborhood of the bifurcating point) is essentially similar to the one in item 7. Note only that the fact that  $M(q_m) \neq 0$  make the eigenvalue-eigenvector equations read

$$\begin{aligned} B_c u_c + \lambda B_l L(i_l)w_l + B_r R(i_r)w_r + B_m M(q_m)w_m + B_j u_j &= 0 \\ \lambda Q_c C(v_c)u_c + Q_l w_l + Q_r w_r + Q_m w_m + Q_u w_u &= 0, \end{aligned}$$

and (15) is now

$$(\text{Re } \lambda) (u_c^* C(v_c)u_c + w_l^* L(i_l)w_l) + w_r^* \frac{R(i_r) + (R(i_r))^T}{2} w_r + w_m^* M(q_m)w_m = 0.$$

But since not only  $C(v_c)$ ,  $L(i_l)$ ,  $R(i_r)$  but also  $M(q_m)$  are positive definite (always at equilibrium points close enough to the bifurcating one), from the assumption  $\text{Re } \lambda \geq 0$  we would derive  $w_m = w_r = 0$  and the reasoning proceeds as in item 7 above to show that all non-vanishing eigenvalues must verify  $\text{Re } \lambda < 0$ . On the other hand, the fact that only one real eigenvalue changes sign (and hence becomes positive) in the transition to

the region where  $M(q_m) < 0$  follows from the TBWP phenomenon itself. This completes the proof.  $\square$

Theorem 4 shows that, in essence, the presence of a VML-loop (in particular of an ML-loop) with the memristor and at least one inductor is the essential configuration yielding a transcritical bifurcation without parameters, which occurs if the memristance  $M(q_m)$  vanishes (and eventually becomes negative) at a given  $q_m^*$ . A simple parallel connection of a memristor and a linear inductor yields this phenomenon, as shown in [17].

Note that Theorem 4 assumes that, except for the memristor, the remaining circuit devices are strictly locally passive. If this assumption is relaxed things become more complicated; we present some results in this context in the forthcoming section.

## 5 Non-passive problems

Consider the circuit displayed in Figure 1. Note that the absence of a (V)ML-loop rules out an application of Theorem 4 in order to characterize an eventual transcritical bifurcation without parameters in this circuit. However, it is easy to check that the series connection of the linear resistor  $R$  and the memristor  $M$  can itself be modelled as a memristor with memristance  $R + M(q)$ . Moreover, provided that  $R + M(q^*) = 0$  at a given  $q^*$  (with  $M'(q^*) \neq 0$ ), the circuit is expected and can be easily shown to undergo a TBWP.

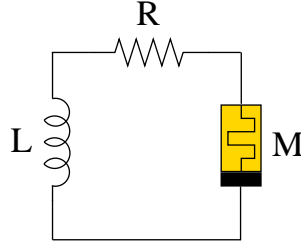


Figure 1: MRL-circuit.

Obviously, for the condition  $R + M(q^*) = 0$  to hold, either both  $R$  and  $M$  must vanish or one of them need become negative. In particular, when  $R$  becomes zero or negative, the problem does not fit the strict passivity assumption for resistors stated in Theorem 4. This means that a wider framework is needed to address this phenomenon in general. Although in its broad generality this is a difficult problem, some remarks in this direction can be provided, as detailed in what follows.

Specifically, we provide below conditions guaranteeing, in non-passive circuits with a single memristor, that the null eigenvalue is indeed a multiple one, generalizing (as detailed later, cf. the Remark after Proposition 1) the framework considered in Theorem



4. We use the notion of both a proper tree and an L-proper tree. Split the branches of a given connected graph  $\mathcal{G}$  in three pairwise disjoint sets  $K_1$ ,  $K_2$  and  $K_3$ , in a way such that  $K_1$  includes no loops and  $K_3$  no cutsets. Then, as a consequence of the matroid structure of the set of acyclic subgraphs of  $\mathcal{G}$  [30], one can guarantee that there exists at least one spanning tree including all branches from  $K_1$  and none from  $K_3$  (an explicit proof can be found e.g. in [7]). Such a spanning tree is called (in general) a *proper tree*. In circuit theory, this term is usually restricted to connected circuits without VC-loops and IL-cutsets, to denote a spanning tree including all voltage sources and capacitors, and neither current sources nor inductors. This notion can be traced back at least to [5]. We will also make use of the (in a certain sense dual) concept of an *L-proper tree*, which is a spanning tree including all voltage sources and inductors, and neither current sources nor capacitors; such a tree exists if and only if the circuit has neither VL-loops nor IC-cutsets. From [38] we borrow the concept of an MR-product; given a spanning tree in a connected circuit with voltage and current sources, capacitors, inductors, resistors and memristors, the *MR-product* of this tree is simply the product of all resistances and memristances in the co-tree branches (namely, the branches that do not belong to the spanning tree), evaluated at equilibrium and setting this product to 1 if all resistors and memristors are actually located in the tree.

**Proposition 1.** *Consider, as in Theorem 4, a circuit with a single memristor displaying an equilibrium point at a given  $(y^*, z^*)$ , with  $y^* = (q_m^*, v_c^*, i_l^*)$  and  $z^* = (i_m^*, i_c^*, v_l^*, i_r^*, v_j^*, i_u^*)$ . Assume that  $C(v_c^*)$  and  $L(i_l^*)$  are non-singular, besides the following.*

1. *The circuit displays no VC-loops, IL-cutsets, VL-loops or IC-cutsets.*
2. *The sum of MR-products in proper trees does not vanish.*
3. *The sum of MR-products in L-proper trees does vanish.*

*Then the algebraic multiplicity of the zero eigenvalue at  $(y^*, z^*)$  is greater than one.*

The proof of this result follows from the results detailed in [38]. First, the absence of VC-loops and IL-cutsets, together with the non-vanishing condition on the MR-product sum in proper trees, guarantees the matrix  $g_z(y^*, z^*)$  to be non-singular, cf. Proposition 3 in [38]. Note that this matrix has now the form

$$g_z(y^*, z^*) = \begin{pmatrix} B_m M(q_m^*) & 0 & B_l & B_r R(i_r^*) & B_j & 0 \\ Q_m & Q_c & 0 & Q_r & 0 & Q_u \end{pmatrix}. \quad (17)$$

Additionally, the null eigenvalue having a multiplicity greater than one is in a way the dual property to the one above, and relies on the structure of the matrix  $K(0)$  from (14), which in the presence of a possibly non-vanishing memristance reads as

$$\begin{pmatrix} 0 & 0 & 0 & -(C(v_c^*))^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(L(i_l^*))^{-1} & 0 & 0 & 0 \\ -B_c & 0 & -B_m M(q_m^*) & 0 & -B_l & -B_r R(i_r^*) & -B_j & 0 \\ 0 & -Q_l & -Q_m & -Q_c & 0 & -Q_r & 0 & -Q_u \end{pmatrix}.$$

Again, this matrix can be checked to be singular if and only if so it is the matrix

$$\begin{pmatrix} B_c & 0 & B_m M(q_m^*) & B_r R(i_r^*) & B_j & 0 \\ 0 & Q_l & Q_m & Q_r & 0 & Q_u \end{pmatrix}, \quad (18)$$

but from Proposition 3 in [38] one can show that, in the absence of VL-loops and IC-cutsets, the latter matrix is singular if and only if the sum of MR-products in L-proper trees does vanish. Details are not difficult and are left to the reader. The above-referred duality property becomes clear in the light of the matrices (17) and (18), which have exactly the same form after a change of reactive devices (capacitors and inductors) and an obvious column reordering.

Proposition 1 opens a way for future research, which should address the remaining conditions from Theorem 3 in order to characterize the TBWP in this wider setting. Note that Proposition 1 does not require any passivity assumption on the circuit characteristic matrices.

**Remark.** In the presence of an VML-loop, as in the setting of Theorem 4, the memristor must by definition belong to the cotree of all L-proper trees (since these must accommodate all voltage sources and inductors);  $M$  is therefore a common factor in all MR-products and therefore the condition  $M = 0$  in Theorem 4 arises naturally.

Proposition 1 above explains, in topological terms, why the bifurcating condition in the circuit of Figure 1 is  $M(q_m^*) + R = 0$ . Just note that the circuit has two L-proper trees, displayed in Figure 2, the cotrees of which amount, respectively, to the memristor  $M$  and the resistor  $R$ ; the sum of products arising in item 3 of Proposition 1 is just  $M(q_m^*) + R$  and for this reason the vanishing of this sum is responsible for a multiple zero eigenvalue supporting the TBWP. However, the scope of this results goes much further than this elementary (pedagogic) example, since the computation of spanning trees is an easily automatable task and therefore the result applies to much more complex circuits. A case of intermediate complexity is discussed below with illustrative purposes.



Figure 2: L-proper trees in the circuit of Figure 1

## A memristive artificial neural network

Memristors provide an excellent framework for the implementation of artificial neural networks. The key reason is that they define a nanometric scale, electrically adaptable

device perfectly suited for implementing neural synapses, emulating (to a certain extent) the STDP mechanism from biological neural systems [39]. A lot of recent literature explores this idea; see e.g. [2, 21, 23, 33, 40, 47] and references therein.

In this context, we analyze below a simplification of an additive model proposed in [47]. We ignore delays and assume that each neuron is defined by a passive RC-connection, and also that the input-output function in each neuron is just implemented by a linear passive resistor. Furthermore, we focus on a problem with just two neurons and assume that the conductivities of three out of four synaptic connections are fixed, in order to concentrate the attention on a bifurcating memristor. This simplified model is depicted in Figure 3 (a); in (b) we arrange the circuit in a more convenient manner for later computations.

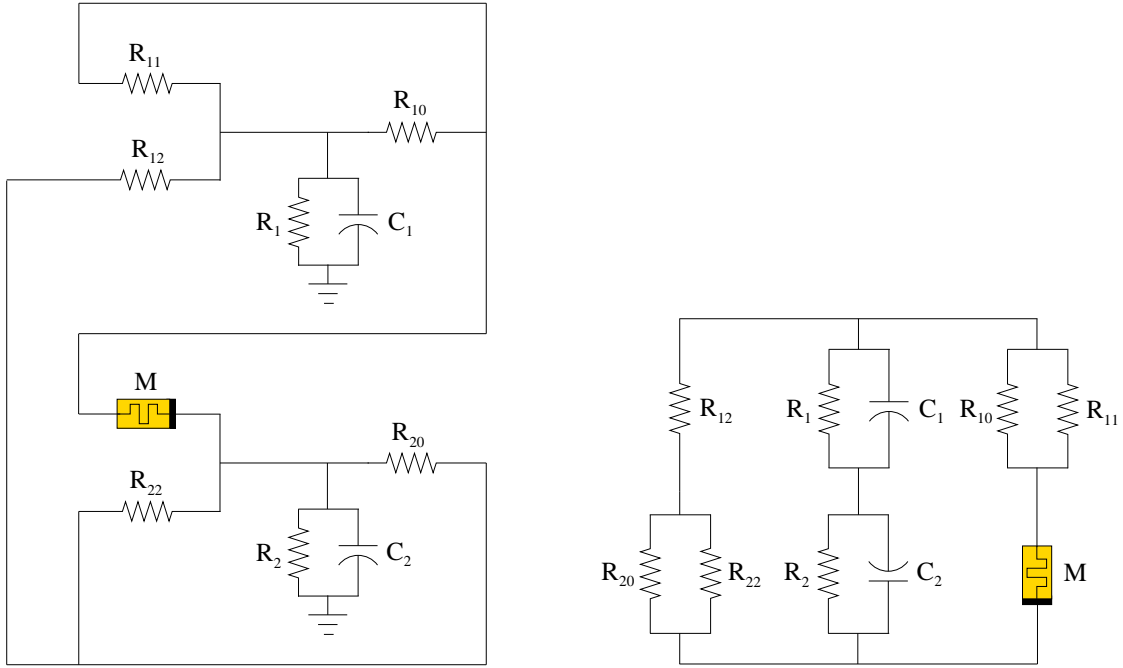


Figure 3: (a) Memristive network.

(b) Alternative circuit description.

Note that, for any charge  $q_m$  in the memristor and null values in the remaining circuit variables one gets an equilibrium point. This simply expresses the presence of a line of equilibria along the  $q_m$ -coordinate axis, and the linearized dynamics exhibits a zero eigenvalue along this axis. Omitting details for the sake of brevity, a state-space model for this circuit indicates that the condition for the zero eigenvalue to be multiple is

$$R_A R_B + (R_1 + R_2) [(R_{20} + R_{22}) R_A + (R_{10} + R_{11}) R_B] = 0, \quad (19)$$

with

$$R_A = M(R_{10} + R_{11}) + R_{10} R_{11}, \quad R_B = R_{12}(R_{20} + R_{22}) + R_{20} R_{22}.$$

This yields the bifurcation value

$$M = \frac{-R_C - R_D}{(R_{10} + R_{11})[R_B + (R_1 + R_2)(R_{20} + R_{22})]}$$

where

$$R_C = R_{10}R_{11}[R_B + (R_1 + R_2)(R_{20} + R_{22})], \quad R_D = R_B(R_1 + R_2)(R_{10} + R_{11}).$$

Actually, for negative values  $M$  smaller than the one above, and provided that all resistances are positive, a transition of an eigenvalue to the positive real semiaxis signals a stability loss due to a TBWP.

Our goal is to explain in topological terms the condition (19) on the resistances and the memristance, which makes this null eigenvalue a multiple one, in order to illustrate the scope of Proposition 1. Note, that in large scale circuits the derivation of a model and therefore the explicit computation of the bifurcation conditions are usually unfeasible, and for this reason one has no option but to resort to circuit-theoretic results such as the one in Proposition 1. To achieve this one needs to compute the set of L-proper trees of the circuit. In our present example there are actually 33 L-proper trees, depicted in Figure 4.

One can check that this set of trees actually explains the bifurcating condition (19), in light of item 3 of Proposition 1. Specifically, the terms responsible for the product  $R_A R_B$  arise from the cotree branches of the trees 5, 8, 11, 12, 15, 18, 23, 28 and 33 in Figure 4; analogously, the terms  $(R_1 + R_2)(R_{20} + R_{22})R_A$  come from the cotree branches of trees 3, 4, 9, 10, 16, 17, 21, 22, 26, 27, 31 and 32, whereas the products  $(R_1 + R_2)(R_{10} + R_{11})R_B$  arise from the cotrees of the trees 1, 2, 6, 7, 13, 14, 19, 20, 24, 25, 29 and 30.

As indicated above, this graph-theoretic characterization of the bifurcation condition is easily scalable to large circuits in which the analytical (model-based) computation of such condition is not feasible.

## 6 Concluding remarks

We have presented in this paper a detailed circuit-theoretic characterization of the transcritical bifurcation without parameters in circuits with one memristor, systematically yielding lines of equilibrium points. To do so, we have developed mathematical statements of the TBWP theorem for explicit ODEs in arbitrary dimension and also for semiexplicit DAEs, which are believed to be of independent interest. This allows for a graph-theoretic analysis of the bifurcation in the circuit context. Future research should provide a complete characterization of this phenomenon in non-passive settings, along the lines discussed in Section 5. Other related bifurcations, such as the Hopf bifurcation without parameters, might be analyzed in similar terms.

## Appendix: digraph matrices

In the formulation of the circuit model (11) we make use of the so-called *loop* and *cutset* matrices defined below. Given a digraph with  $m$  edges,  $n$  nodes and  $k$  connected components, choose an orientation in every loop and define componentwise the *loop matrix*  $\tilde{B}$  as  $(b_{ij})$ , with

$$b_{ij} = \begin{cases} 1 & \text{if edge } j \text{ is in loop } i \text{ with the same orientation} \\ -1 & \text{if edge } j \text{ is in loop } i \text{ with the opposite orientation} \\ 0 & \text{if edge } j \text{ is not in loop } i. \end{cases}$$

This matrix has rank  $m - n + k$ , and a *reduced loop matrix*  $B$  is any  $((m - n + k) \times m)$ -submatrix of  $\tilde{B}$  with full row rank.

The dual concept is that of a reduced cutset matrix. Recall that a set  $K$  of edges in a digraph is a *cutset* if the removal of  $K$  increases the number of connected components, and  $K$  is minimal with respect to this property, that is, retaining one or more edges from  $K$  keeps the number of components invariant. All the edges of a cutset may be shown to connect the same pair of connected components which result from the cutset deletion, and this allows one to define the orientation of a cutset, say from one of these components towards the other. This makes it possible to define the cutset matrix  $\tilde{Q} = (q_{ij})$  as

$$q_{ij} = \begin{cases} 1 & \text{if edge } j \text{ is in cutset } i \text{ with the same orientation} \\ -1 & \text{if edge } j \text{ is in cutset } i \text{ with the opposite orientation} \\ 0 & \text{if edge } j \text{ is not in cutset } i. \end{cases}$$

Now the rank of  $\tilde{Q}$  is  $n - k$ , and a *reduced cutset matrix*  $Q \in \mathbb{R}^{(n-k) \times m}$  is obtained by choosing any set of  $n - k$  linearly independent rows of  $\tilde{Q}$ .

## References

- [1] A. Adamatzky and L. O. Chua (eds.), *Memristor Networks*, Springer, 2014.
- [2] S. P. Adhikari, C. Yang, H. Kim and L. O. Chua, Memristor bridge synapse-based neural network and its learning, *IEEE Trans. Neural Networks and Learning Systems* **23** (2012) 1426-1435.
- [3] B. Andrásfai, *Introductory Graph Theory*, Akadémiai Kiadó, Budapest, 1977.
- [4] B. Aulbach, *Continuous and Discrete Dynamics near Manifolds of Equilibria*, Lect. Note Math. 1058, Springer-Verlag, 1984.
- [5] T. R. Bashkow, The  $A$  matrix, new network description, *IRE Trans. Circuit Theory* **4** (1957) 117-119.
- [6] B. Bollobás, *Modern Graph Theory*, Springer-Verlag, 1998.

- [7] D. P. Brown, Derivative-explicit differential equations for RLC graphs, *Journal of the Franklin Institute* **275** (1963) 503-514.
- [8] P. R. Bryant, The explicit form of Bashkow's  $A$  matrix, *IRE Trans. Circuit Theory* **9** (1962) 303-306.
- [9] J. Carr, *Applications of Center Manifold Theory*, Springer-Verlag, 1981.
- [10] L. O. Chua, Memristor – The missing circuit element, *IEEE Trans. Circuit Theory* **18** (1971) 507-519.
- [11] L. O. Chua, Dynamic nonlinear networks: state-of-the-art, *IEEE Trans. Circuits and Systems* **27** (1980) 1059-1087.
- [12] M. Di Ventra, Y. V. Pershin and L. O. Chua, Circuit elements with memory: memristors, memcapacitors and meminductors, *Proc. IEEE* **97** (2009) 1717-1724.
- [13] D. Estévez-Schwarz and C. Tischendorf, Structural analysis of electric circuits and consequences for MNA, *Intl. J. Circuit Theory Appl.* **28** (2000) 131-162.
- [14] B. Fiedler, S. Liebscher, and J. C. Alexander, Generic Hopf bifurcation from lines of equilibria without parameters: I. Theory, *J. Differential Equations* **167** (2000) 16-35.
- [15] B. Fiedler and S. Liebscher, Generic Hopf bifurcation from lines of equilibria without parameters: II. Systems of viscous hyperbolic balance laws, *SIAM J. Math. Anal.* **31** (2000) 1396-1404.
- [16] B. Fiedler, S. Liebscher, and J. C. Alexander, Generic Hopf bifurcation from lines of equilibria without parameters: III. Binary oscillations, *Internat. J. Bifur. Chaos* **10** (2000) 1613-1622.
- [17] I. García de la Vega and R. Riaza, Bifurcation without parameters in circuits with memristors: A DAE approach, *Proc. AIMS'2014*, pp. 340-348, 2015.
- [18] M. Günther and U. Feldmann, CAD-based electric-circuit modeling in industry. I: Mathematical structure and index of network equations, *Surv. Math. Ind.* **8** (1999) 97-129.
- [19] M. Itoh and L. O. Chua, Memristor oscillators, *Internat. J. Bifur. Chaos* **18** (2008) 3183-3206.
- [20] S. Iwata, M. Takamatsu and C. Tischendorf, Tractability index of hybrid equations for circuit simulation, *Mathematics of Computation* **81** (2012) 923-939.
- [21] S. H. Jo, T. Chang, I. Ebong, B. B. Bhadviya, P. Mazumder and W. Lu, Nanoscale memristor device as synapse in neuromorphic systems, *Nano Letters* **10** (2010) 1297-1301.

- [22] O. Kavehei, A. Iqbal, Y. S. Kim, K. Eshraghian, S. F. Al-Sarawi and D. Abbott, The fourth element: characteristics, modelling and electromagnetic theory of the memristor, *Proc. R. Soc. A* **466** (2010) 2175-2202.
- [23] H. Kim, M. P. Sah, C. Yang, T. Roska and L. O. Chua, Memristor bridge synapse, *Proc. IEEE* **100** (2012) 2061-2070.
- [24] R. Lamour, R. März and C. Tischendorf, *Differential-Algebraic Equations. A Projector Based Analysis*, Springer, 2013.
- [25] S. Liebscher, *Bifurcation without Parameters*, Springer, 2015.
- [26] M. Messias, C. Nespoli and V. A. Botta, Hopf bifurcation from lines of equilibria without parameters in memristors oscillators, *Internat. J. Bifur. Chaos* **20** (2010) 437-450.
- [27] G. J. Minty, Monotone networks, *Proc. Royal Soc. London A* **257** (1960) 194-212.
- [28] B. Muthuswamy, Implementing memristor based chaotic circuits, *Internat. J. Bifur. Chaos* **20** (2010) 1335-1350.
- [29] B. Muthuswamy and L. O. Chua, Simplest chaotic circuit, *Internat. J. Bifur. Chaos* **20** (2010) 1567-1580.
- [30] J. Oxley, *Matroid Theory*, Oxford Univ. Press, 1992.
- [31] K. J. Palmer, Linearization near an integral manifold, *J. Math. Anal. Appl.* **51** (1975) 243-255.
- [32] L. Perko, *Differential Equations and Dynamical Systems*, Springer, 2001.
- [33] Y. V. Pershin and M. Di Ventra, Experimental demonstration of associative memory with memristive neural networks, *Neural Networks* **23** (2010) 881-886.
- [34] R. Riaza, *Differential-Algebraic Systems. Analytical Aspects and Circuit Applications*, World Scientific, 2008.
- [35] R. Riaza, Nondegeneracy conditions for active memristive circuits, *IEEE Trans. Circuits and Systems - II* **57** (2010) 223-227.
- [36] R. Riaza, Stability loss in quasilinear DAEs by divergence of a pencil eigenvalue, *SIAM J. Math. Anal.* **41** (2010) 2226-2245.
- [37] R. Riaza, Manifolds of equilibria and bifurcations without parameters in memristive circuits, *SIAM J. Appl. Math.* **72** (2012) 877-896.
- [38] R. Riaza, Cyclic matrices of weighted digraphs, *Discrete Appl. Math.* **160** (2012) 280-290.

- [39] T. Serrano-Gotarredona, T. Masquelier, T. Prodromakis, G. Indiveri and B. Linares-Barranco, STDP and STDP variations with memristors for spiking neuromorphic learning systems, *Frontiers in Neuroscience* **7** (2013) 2013-2.
- [40] D. Soudry, D. Di Castro, A. Gal, A. Kolodny and S. Kvatinsky, Memristor-based multilayer neural networks with online gradient descent training, *IEEE Trans. Neural Networks and Learning Systems* **26** (2015) 2408-2421.
- [41] D. B. Strukov, G. S. Snider, D. R. Stewart and R. S. Williams, The missing memristor found, *Nature* **453** (2008) 80-83.
- [42] M. Takamatsu, Structural characterization of hybrid equations with tractability index at most two, *J. Math. Anal. Appl.* **411** (2014) 639-651.
- [43] R. Tetzlaff (ed.), *Memristors and Memristive Systems*, Springer, 2014.
- [44] C. Tischendorf, Topological index calculation of DAEs in circuit simulation, *Surv. Math. Ind.* **8** (1999) 187-199.
- [45] C. Tischendorf, Coupled systems of differential algebraic and partial differential equations in circuit and device simulation. Modeling and numerical analysis, Habilitationsschrift, Humboldt-Univ. Berlin, 2003.
- [46] J. Vandewalle and L. O. Chua, The colored branch theorem and its applications in circuit theory, *IEEE Trans. Circuits and Systems* **27** (1980) 816-825.
- [47] S. Wen, Z. Zeng and T. Huang, Exponential stability analysis of memristor-based recurrent neural networks with time-varying delays, *Neurocomputing* **97** (2012) 233-240.
- [48] S. Wiggins, *Introduction to Applied Dynamical Systems and Chaos*, Springer, 2003.



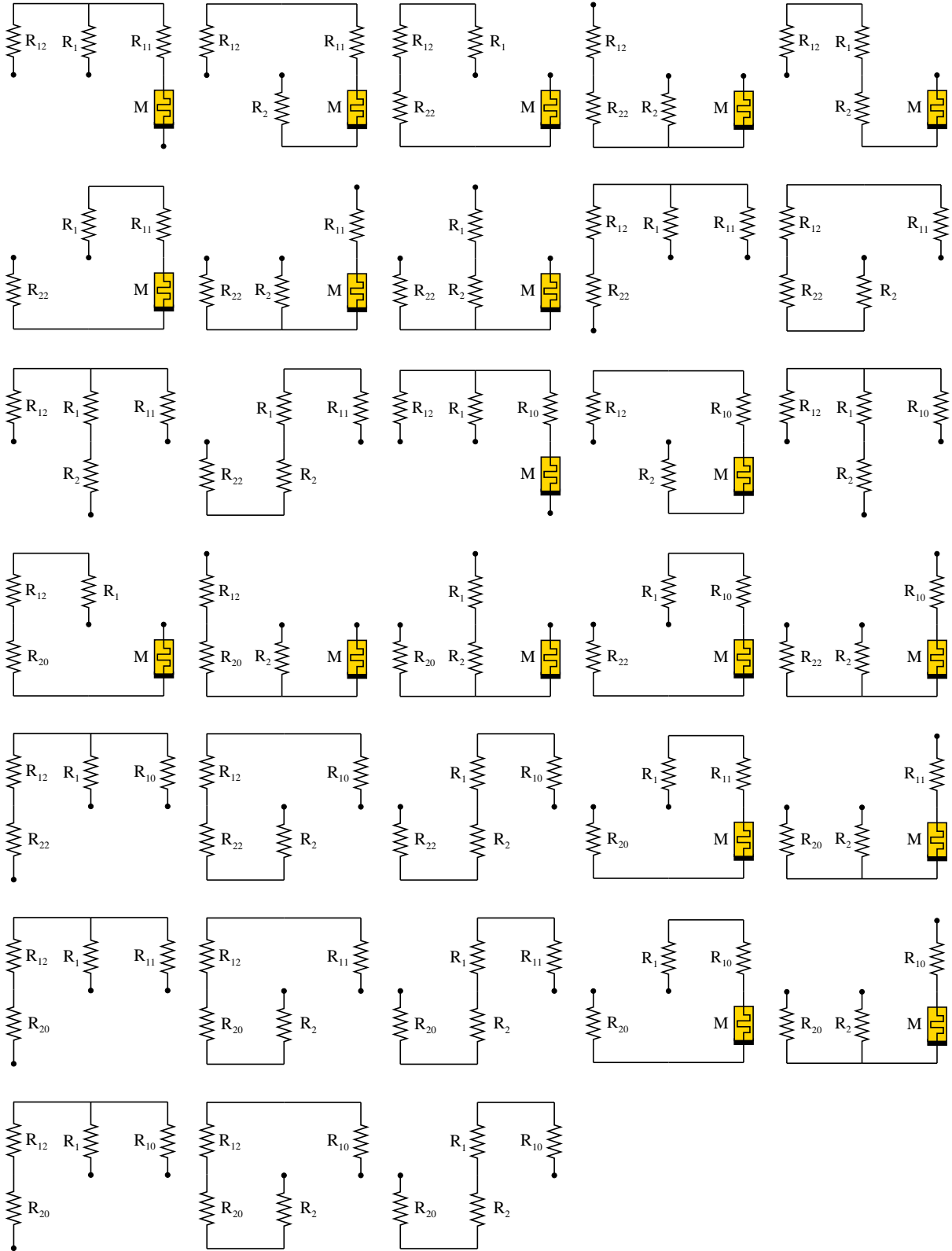


Figure 4: L-proper trees for the circuit in Figure 3.